San Diego Math Circle
Fermat Class, 09/27/2008
Solutions, Revision 1

Fill in the missing numbers in the following patterns:

1. $1, 2, 3, 4, \boxed{5}, \boxed{6}, \boxed{7}, 8, 9, ...$
2. $2, 5, 8, \boxed{11}, \boxed{14}, \boxed{17}, 20, 23, ...$
3. $1, 2, 4, \boxed{8}, \boxed{16}, \boxed{32}, 64, 128, ...$
4. $10, 8, 6, \boxed{4}, \boxed{2}, 0, -2, -4, \boxed{-6}, \boxed{-8}, \boxed{-10}, ...$
5. $128, \boxed{64}, \boxed{32}, -16, 8, -4, \boxed{2}, \boxed{-1}, ...$
6. $64, \boxed{96}, 144, 216, \boxed{324}, \boxed{486}, 729, ...$

Compute the following sums:

For more info on counting the numbers in lists, see Richard's first two videos here:
`http://www.artofproblemsolving.com/Alcumus/Videos/VideoList.php`
Note that you will need to be logged into the AoPS website for the above link to work, otherwise you will be taken to the main AoPS page.

1. $1 + 2 + 3 + ... + 199 + 200$
   Note that there are 200 numbers (or 100 pairs). The sum for each pair is $1 + 200 = 201$, so we can compute the sum by multiplying: $201 \times 100 = \boxed{20100}$.

2. $3 + 6 + 9 + ... + 597 + 600$
   Note that there are 200 numbers (or 100 pairs). The sum for each pair is $3 + 600 = 603$, so we can compute the sum by multiplying: $603 \times 100 = \boxed{60300}$.

   Alternatively, since each number in this sum is exactly 3 times the corresponding number in the first sum above, the sum should also be exactly 3 times the sum in the first problem: $3 \times 20100 = \boxed{60300}$.

3. $2 + 5 + 8 + ... + 596 + 599$
   Note that there are 200 numbers (or 100 pairs). The sum for each pair is $2 + 599 = 601$, so we can compute the sum by multiplying: $601 \times 100 = \boxed{60100}$.

   Alternatively, since each number in this sum is exactly 1 less than the corresponding number in the sum above, the sum should be exactly 200 (1 less for each of 200 numbers) less than the sum above : $60300 - 200 = \boxed{60100}$.

4. $-20 + -16 + -12 + .. + 144 + 148 + 152$
   Note that there are 44 numbers (or 22 pairs). The sum for each pair is $-20 + 152 = 132$, so we can compute the sum by multiplying: $132 \times 22 = \boxed{2904}$.

   Alternatively, if we group the terms appropriately, $(-20 + -16 + -12 + ... + 12 + 16 + 20) + (24 + 28 + ... + 152)$, the numbers in the first batch cancel out, and we are left with $24 + 28 + ... + 152$. This sum has 33 numbers (or 16.5 pairs). The sum for each pair is $24 + 152 = 176$, so we can compute the sum by multiplying: $176 \times 16.5 = \boxed{2904}$.

Compute the following sums:

For a finite geometric series, the formula for the sum is: $a_0 \times \frac{1-r^{n+1}}{1-r}$, where a_0 is the first term, r is the common ratio, n is the number of elements.
For an infinite geometric series, the formula for the sum (if $|r| < 1$) is: $\frac{a_0}{1-r}$, where a_0 is the first term and r is the common ratio.

1. $1 + 2 + 4 + .. + 256 + 512$
   Since this is a geometric series with a common ratio of 2 and 9 elements, the sum is equal to $\frac{1-2^{9+1}}{1-2} = \frac{1-1024}{-1} = \boxed{1023}$.

2. $1 + -3 + 9 + ... + -243 + 729$
   Since this is a geometric series with a common ratio of -3 and 6 elements, the sum is equal to $\frac{1-(-3)^{6+1}}{1-(-3)} = \frac{1+2187}{4} = \boxed{547}$.

3. $1 + \frac{1}{2} + \frac{1}{4} + ...$
   Since this an infinite geometric series with a common ratio of $\frac{1}{2}$, the sum is equal to $\frac{1}{1-\frac{1}{2}} = \boxed{2}$.

4. $1 + -\frac{1}{3} + \frac{1}{9} + -\frac{1}{27} + ...$
   Since this an infinite geometric series with a common ratio of $-\frac{1}{3}$, the sum is equal to $\frac{1}{1-(-\frac{1}{3})} = \boxed{\frac{3}{4}}$.

Convert the following repeating decimals into fractions:

1. $0.\overline{3} = 0.3333...$
   Converting this into an infinite geometric series: $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ... = \boxed{\frac{1}{3}}$.

2. $0.\overline{6} = 0.6666...$
   Converting this into an infinite geometric series: $0.\overline{6} = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + ... = \boxed{\frac{2}{3}}$.

3. $0.\overline{9} = 0.9999...$
   Converting this into an infinite geometric series: $0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ... = \boxed{1}$.

4. $0.\overline{152} = 0.152152152...$
   Converting this into an infinite geometric series:
   $0.\overline{152} = \frac{152}{1000} + \frac{152}{1000000} + \frac{152}{1000000000} + ... = \boxed{\frac{152}{999}}$.

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5. $0.7\overline{2} = 0.72222...$
   $0.7\overline{2} = \frac{7}{10} + \frac{2}{100} + \frac{2}{1000} + \frac{2}{10000} + ...$ which is $\frac{7}{10}$ plus an infinite geometric series. The sum is then $\frac{7}{10} + \frac{2}{90} = \boxed{\frac{13}{18}}$.

6. $0.23\overline{45} = 0.23454545...$
   $0.23\overline{45} = \frac{23}{100} + \frac{45}{10000} + \frac{45}{1000000} + \frac{45}{100000000} + ...$ which is $\frac{23}{100}$ plus an infinite geometric series. The sum is then $\frac{23}{100} + \frac{45}{9900} = \frac{2322}{9900} = \boxed{\frac{129}{550}}$.